REVISTA DE CIENCIAS, Vol. 6, No. 3 de 2015.

# Modos de Disipación Magnética en la Corteza de Estrellas de Neutrones

## Magnetic Diffusion Modes in Neutron Star Crusts

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Aceptado Diciembre de 2014; Publicado en línea Marzo de 2015.

ISSN 2256-3830.

### Resumen

Las estrellas de neutrones son objetos compactos remanentes en las explosiones de supernovas. Observaciones astronómicas sugieren que el campo magnético superficial de estas estrellas decae en escalas largas de tiempo, proceso que debe ser mediado por efectos disipativos tales como la difusión lineal Ohmica. Aunque es bien sabido que las escalas de tiempo de la difusión Ohmica son mucho más largas que la edad del Universo, procesos no lineales como la difusión ambipolar o el efecto Hall pueden generar estructuras de pequeña escala que acortarían las escalas de tiempo [1]. En este artículo se ilustra el cálculo de los modos de difusión Ohmica confinados en la corteza esférica de estrellas de neutrones bajo simetría axial (2D). La solución de las ecuaciones diferenciales parciales del modelo se basa en un método espectral que expande las funciones angulares en polinomios de Legendre mientras que la parte radial y temporal se resuelve por separación de variables.

***Palabras Claves*:** Estrellas de Neutrones, Campos Magnéticos.

### Abstract

Neutron stars are compact objects remaining of supernova explosions. Astronomical observations suggest that surface star magnetic fields decay over long dissipative time scales. Although it is well known that the diffusive time scales are much longer than the age of the universe, non linear processes such as ambipolar diffusion or Hall effect can generate small-scale structures that shorten the time scales [1]. In this paper we calculate the magnetic diffusion modes confined in spherical neutron star crusts with axial symmetry (2D). The solution of the partial differential equations is based on a spectral method that expands the angular functions in Legendre polynomials while the radial and temporal part are solved by separation of variables.

***Keywords*:** Neutron Stars, Magnetic Fields.

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## 1. Introduction

Astronomical observations of neutron stars show that the age of the star has a relation respect to the magnetic-field strength. Young neutron stars have stronger magnetic fields than old stars. Also it is argued that the decay of ultra strong magnetic field in magnetars are the main source of their Xray luminosity since these objects appear to radiate substantially more power than that available from their rotational energy loss. As a consequence of the high electrical conductivity in the neutron start crust, the time scales of diffusive processes are longer than the age of Universe, however some non - linear processes such as the Hall effect can generate structures with small length scales which shorten the time scales of these diffusive modes, thus, the magnetic field can decay in reasonable astronomical time scales. In this paper we calculate the magnetic energy of different Ohmic (or diffusive) confined modes and show that these energies decay in some characteristic time scales.

## 2. Methodology

### 2.1. *Evolution Equation*

The idea is to solve the equation for the evolution of the solenoidal magnetic field $\overrightarrow{B}(r,\theta,t)$ in a spherical shell $(r_i < r < r_o)$ under the effect the linear Ohmic diffusion with the electrical conductivity $\sigma_0$ and the electron number density $n_e$ both spatially uniform functions:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}^2 \vec{B}, \tag{1}$$

where the time is normalized respect to $\tau_{ohm} = \frac{4\pi\sigma_o L^2}{c^2}$ with $L$ a characteristic length of the system. Note that this equation is consistent with the condition $\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$, actually we used this condition in the derivation of the Ohmic diffusion term in Eq.(1) by writing $-\overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{B}) = \overrightarrow{\nabla}^2 \overrightarrow{B}$. Thus, if we take the divergence of Eq.(1) we obtain that $(\frac{\partial}{\partial t})(\overrightarrow{\nabla} \cdot \overrightarrow{B}) = 0$, so, the magnetic field conserves its solenoidal property $(\overrightarrow{\nabla} \cdot \overrightarrow{B}) = 0$ along the evolution.

For this task we use a decomposition of field in toroidal and poloidal parts developed in [5] where the magnetic field is written in terms of two unknown scalar functions $g(r,\theta,t)$ and $h(r,\theta,t)$ as:

$$\overrightarrow{B} = \overrightarrow{\nabla} \times (g\hat{\overrightarrow{r}}) + \overrightarrow{\nabla} \times \left[\overrightarrow{\nabla} \times (h\hat{\overrightarrow{r}})\right], \tag{2}$$

with $\hat{\overrightarrow{r}}$ is the radial unit vector and $g$, $h$ expanded in Legendre polynomials as:

$$g(r,\theta,t) = \sum_{l=1}^{L_B} g_l(r,t) P_l(\cos\theta), \tag{3}$$

$$h(r,\theta,t) = \sum_{l=1}^{L_B} h_l(r,t) P_l(\cos\theta), \tag{4}$$

with $L_B$ the truncation of the series. Taking the $r$ component of Eq.(2) and the $r$ component of the curl of Eq.(2) after using Eq(1) and Eqs(3-4) we obtain the next equations for $h_l(r,t)$ and $g_l(r,t)$, with the radial operator $L_l = \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}$:

$$\sum_l \frac{l(l+1)}{r^2} \left[\frac{\partial}{\partial t} - L_l\right] h_l(r,t) P_l(\cos\theta) = 0, \tag{5}$$

$$\sum_l \frac{l(l+1)}{r^2} \left[\frac{\partial}{\partial t} - L_l\right] g_l(r,t) P_l(\cos\theta) = 0. \tag{6}$$

2.2. *Linear Ohmic decay modes*

The ohmic decay modes are given by $g(r,\theta,t) = g_l(r,t)P_l(\cos\theta)$ and $h(r,\theta,t) = h_l(r,t)P_l(\cos\theta)$, thus we obtain the linear equations for the radial part of these modes for each $(l)$ mode independently as:

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right] h_l(r,t) = 0 \quad ; \quad \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right] g_l(r,t) = 0. \tag{7}$$

Also note that both equations have the same functional structure, so it is enough to focus in the first one for $h_l(r)$ and the same conclusions are valid at the end for $g_l(r)$. Thus, we look for separable solutions $h_l(r,t) = R_l(r)T_l(t)$ which is inserted in Eq.(7) to obtain:

$$T_l(t) = e^{-k_l^2 t} = e^{\frac{-t}{\tau_l}} \quad ; \quad r^2\frac{d^2R_l}{dr^2} - \left[l(l+1) - (k_l r)^2\right] R_l = 0, \tag{8}$$

where $k_l^2$ is a separation constant that fixes the decay timescales for each mode as $\tau_l = k_l^{-2}$. The different $k_l$ need to be determined by the the boundary conditions on $R_l$ or its derivatives. Starting from the *Spherical Bessel differential equation* we obtain the solution of the Eq.(8) as a linear combination of the spherical Bessel functions times a factor $-r$: $R_l(r) = -r\left[C_l j_l(k_l r) + D_l y_l(k_l r)\right]$. Thus, the ohmic decay modes are: $h(r,\theta,t) = -r\left[C_l j_l(k_l r) + D_l y_l(k_l r)\right] P_l(\cos\theta)e^{\frac{-t}{\tau_l}}$ and $g(r,\theta,t) = -r\left[E_l j_l(k_l r) + F_l y_l(k_l r)\right] P_l(\cos\theta)e^{\frac{-t}{\tau_l}}$. With $C_l$, $D_l$, $E_l$, $F_l$ constants that need to be fixed by the boundary conditions.

2.3. *Zero-Boundary conditions*

From Eq.(2) we can obtain the explicit form of the components of the magnetic field as:

$$B_r = \frac{1}{r\sin\theta}\left[-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right)\right] \quad ; \quad B_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h}{\partial\theta}\right) \quad ; \quad B_\phi = -\frac{1}{r}\left(\frac{\partial g}{\partial\theta}\right). \tag{9}$$

If we have a confined magnetic field we must require at the inner and outer surfaces $B_r(r_i,\theta,t) = B_r(r_o,\theta,t) = 0$. From Eq.(9), this implies at these surfaces the conditions $\left(\frac{\partial h}{\partial\theta}\right)|_{r_i,r_o,\theta,t} = 0$. Thus $h$ needs to be a constant function respect to $\theta$ at the inner and outer surfaces defined by $r_i$ and $r_o$ respectively.

Therefore a simple choice for the boundary conditions are the "zero-boundary" conditions: $h(r_i,\theta,t) = h(r_o,\theta,t) = 0$. The boundary conditions over $g$ are less obvious from the physical point of view and they depend essentially on the structure of $B_\phi$ at the inner and outer surfaces. However and as starting simple point we can set for $g$ the same kind of "zero-boundary" conditions. Therefore: $g(r_i,\theta,t) = g(r_o,\theta,t) = 0$. Normalizing respect to the star radius, with $r_o = 1$. The above boundary conditions lead to:

$$C_l j_l(k_l r_i) + D_l y_l(k_l r_i) = 0; \quad C_l j_l(k_l r_o) + D_l y_l(k_l r_o) = 0 \tag{10}$$

$$E_l j_l(k_l r_i) + F_l y_l(k_l r_i) = 0; \quad E_l j_l(k_l r_o) + F_l y_l(k_l r_o) = 0. \tag{11}$$

Combining Eqs(10-11) we obtain a trascendental equation for $k_l$ :

$$y_l(k_l)j_l(k_l r_i) - y_l(k_l r_i)j_l(k_l) = 0. \tag{12}$$

For a given $l$ we index the diferent solutions of Eq(12) by the index $(n = 1,2,3,\cdots)$, like $k_l^n$. For instance, for a given $l$ the number $k_l^1$ represents the fundamental mode, i.e., that mode with the largest timescale $\tau_1^1 = (k_1^1)^{-2}$ which "survives" in the long-term evolution. Also, by combining these equations we obtain the ratios: $\frac{C_l^n}{D_l^n} = \frac{E_l^n}{F_l^n} = -\frac{y_l(k_l^n)}{j_l(k_l^n)}$. For $r_i = 0.75$ we obtain numerically for $l = 1,2,3,4$ the corresponding modes given by $k_1^1 = 12.6707$, $k_2^1 = 12.8768$, $k_3^1 = 131798$, $k_4^1 = 13.5732$. The corresponding ratios are: $\frac{C_1^1}{D_1^1} = \frac{E_1^1}{F_1^1} = -0.185172$, $\frac{C_2^1}{D_2^1} = \frac{E_2^1}{F_2^1} = 1.65544$, $\frac{C_3^1}{D_3^1} = \frac{E_3^1}{F_3^1} = -1.8269$, $\frac{C_4^1}{D_4^1} = \frac{E_4^1}{F_4^1} = -0.179258$.

## 3. Result

In the next figure we plot the evolution of the logarithm of the magnetic energies for two modes $l$. The logarithm of the analytical solution for the energy of each Ohmic mode scales as $\frac{-2t}{\tau_l^n}$, for long times the energy of all modes have decayed significantly. Note that for the "zero-boundary conditions" we are using both toroidal and poloidal energies of the Ohmic modes decay at the same timescale $\tau_l^n$.

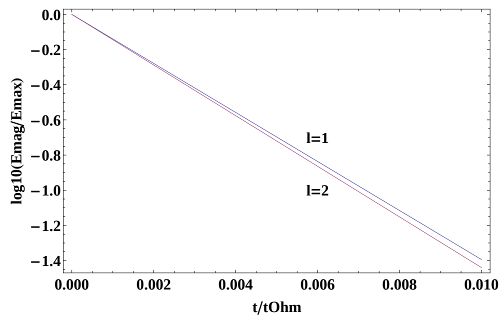


Fig. 1. Evolution of the Magnetic Energy of the modes $(l = 1)$ and $l = 2$, for each case $n = 1$.

## 4. Conclusions

We solved a boundary value problem to study the evolution of the linear magnetic diffusion modes in the crust of a neutron star. Our model was based in the equations of the macroscopic electrodynamics considering an electron fluid in the crust flowing through a nuclei lattice. We used a decomposition of the field in poloidal and toroidal components and we found the evolution of the energy for the corresponding magnetic modes confirming the exponential decay of a characteristic diffusive processes. Although the timescales of these modes are in general much longer than the age of the Universe these scales can be shorten by other non-linear processes such as Hall drift and Ambipolar Diffusion.

## 5. Acknowledgements

The authors acknowledge Andreas Reisenegger, Juan Valdivia and Pablo Marchant for useful scientific conversations.

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