REVISTA DE CIENCIAS, Vol. 5 , No. 1 de 2014.

Nonautonomous multisoliton solutions of the 1D Gross-Pitaevskii equation with time-dependent coefficients

Soluciones multisolitónicas no autónomas de la ecuación unidimensional de Gross-Pitaevskii con coeficientes dependientes del tiempo

Alejandro Pérez Riascos [a] ⋆ , Alvaro Rugeles Pérez [b]

[a] *Instituto de Física, Universidad Nacional Autónoma de México, México, D.F.*
[b] *Departamento de Física, Universidad de Nariño, Pasto.*

Aceptado Octubre; Publicado en línea Noviembre.

ISSN 2256-3830.

Abstract

In this paper we investigate bright multisoliton solutions of the one-dimensional Gross-Pitaevskii equation with a parabolic potential, a time-dependent nonlinearity, and a term related to gain or loss. Analytical N-soliton solutions are obtained by using the relation between the Gross-Pitaevskii equation with time dependent coefficients and the standard nonlinear Schrödinger equation. In particular, we study interactions between snake solitons.

***Keywords*:** Gross-Pitaevskii equation, nonautonomous solitons.

Resumen

En este trabajo se estudian multisolitones brillantes de la ecuación de Gross-Pitaevskii con un potencial parabólico, una no linealidad dependiente del tiempo y un término relacionado con ganancias o pérdidas en un condensado. Se obtienen soluciones analíticas con N solitones utilizando la relación existente entre la ecuación de Gross-Pitaevskii con coeficientes dependientes del tiempo y la ecuación no lineal de Schrödinger. Se hace un estudio particular de los solitones serpiente y sus interacciones.

***Palabras Claves*:** Ecuación de Gross-Pitaevskii, solitones no autónomos.

1. Introduction

Soliton theory describes a class of nonlinear wave propagation phenomena appearing as a result of the balance between nonlinearity and dispersion or diffraction in a system [1]. Solitons have stimulated research in different branches of physics, mathematics and computation science [1, 2, 3]. The ending "on" is generally used to describe elementary particles, and this word was introduced to emphasize the most remarkable feature of these solitary waves. This means that the energy can propagate in the localized form and that the solitary waves emerge from the interaction

⋆ aaappprrr@gmail.com

completely preserved in form and speed with only a phase shift [4].

The classical soliton concept was developed for nonlinear and dispersive systems that have been autonomous; namely, time has only played the role of the independent variable and has not appeared in the coefficients or explicitly in the nonlinear equation that describes the dynamics of the system. However, in experiments and in real situations, it is common that the system is modeled by time-dependent external forces, terms related to losses, and propagation in non-uniform media. In this context are studied nonautonomous solitons, the dynamics of these localized pulses is described by nonlinear differential equations with time-dependent coefficients [4, 5, 6, 7].

Recently much effort has been given to the study of solitons in Bose-Einstein condensates with time dependent control parameters. This phenomenon is modeled by the Gross-Pitaevskii (GP) equation with coefficients that evolve with the time. In the reference [7] is presented a detailed study of its nonautonomous soliton solutions for cases in which the coefficients are related by means of a Riccati type equation. In this work is studied an extension of these results in order to analyze multisoliton solutions.

This paper is organized as follows: in the section 2 we present the mathematical model used in [7] to establish a relation between the Gross-Pitaevskii equation with time-dependent coefficients and the nonlinear Schrödinger (NLS) equation. In section 3 we use an algebraic system of equations that allows to find bright multisoliton solutions of the NLS equation. In section 4 bright multisoliton solutions of the 1D Gross-Pitaevskii equation with time-dependent parameters are investigated. In particular, the cases of two and three snake solitons are discussed. Finally, we present the conclusion and complementary material.

2. Mathematical Model: The 1D Gross-Pitaevskii equation with time-dependent coefficients

The main interest of this paper is to study the interaction of nonautonomous soliton solutions of the equation:

$$i\frac{\partial\Phi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\Phi(x,t)}{\partial x^2} - R(t)|\Phi(x,t)|^2\Phi(x,t) + \frac{\Omega^2(t)}{2}x^2\Phi(x,t) + i\frac{\gamma(t)}{2}\Phi(x,t). \tag{1}$$

Equation (1) is obtained from the Gross-Pitaievskii equation and describes the dynamics of a Bose-Einstein condensate in a parabolic potential with a frequency $\Omega(t)$, time-dependent nonlinearity $R(t)$, and a term $\gamma(t)$ related to gain or loss of atoms in the condensate [7].

A first transformation of (1) combines the term $\frac{\partial\Phi(x,t)}{\partial t}$ with the part modeled by $\gamma(t)$, using [5, 7]:

$$\Phi(x,t) = \exp\left[\frac{1}{2}\int\gamma(t)dt\right]Q(x,t), \tag{2}$$

the equation (1) requires:

$$i\frac{\partial Q(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2 Q(x,t)}{\partial x^2} - \tilde{R}(t)|Q(x,t)|^2Q(x,t) + \frac{\Omega^2(t)}{2}x^2Q(x,t), \tag{3}$$

where the function $\tilde{R}(t)$ is given by $\tilde{R}(t) = \exp\left[\int\gamma(t)dt\right]R(t)$.

The expression (3) is a particular form in a family of integrable nonautonomous NLS equations [4, 5]. In terms of the transformation [4, 5, 7]:

$$Q(x,t) = r(x,t)e^{i\theta(x,t)}q(X,T), \tag{4}$$

the equation (3) is mapped to the standard one-dimensional nonlinear Schrödinger equation:

$$i\frac{\partial q(X,T)}{\partial T} + \frac{1}{2}\frac{\partial^2 q(X,T)}{\partial X^2} + |q(X,T)|^2q(X,T) = 0, \tag{5}$$

where the functions $r(t)$, $\theta(x,t)$, $X(x,t)$, $T(t)$ are defined by the relations [7]:

$$r^2(t) = 2r_0^2\tilde{R}(t), \tag{6}$$

$$\theta(x,t) = -\frac{\tilde{R}_t}{2\tilde{R}}x^2 + 2br_0^2\tilde{R}x - 2b^2r_0^4\int\tilde{R}^2(t)dt, \tag{7}$$

$$X(x,t) = 2r_0\tilde{R}x - 4br_0^3\int\tilde{R}^2(t)dt, \tag{8}$$

$$T(t) = 2r_0^2\int\tilde{R}^2(t)dt. \tag{9}$$

Here b, r_0 are constants and $\tilde{R}(t)$, $\Omega(t)$ satisfy the condition:

$$\frac{d}{dt}\left(\frac{\tilde{R}_t}{\tilde{R}}\right) - \left(\frac{\tilde{R}_t}{\tilde{R}}\right)^2 - \Omega^2(t) = 0. \tag{10}$$

where $\tilde{R}_t = \frac{d\tilde{R}(t)}{dt}$. Hence, transformations (2) and (4) map the GP equation (1) to the NLS equation (5) only if $\tilde{R}(t)$ and $\Omega(t)$ are related by the condition (10) which is a Riccati type equation for $\frac{\tilde{R}_t}{\tilde{R}}$.

Reference [7] presents a detailed study of different analytical solutions of the equation (10) and discusses the characteristics of one soliton solutions of the equation (1). In the appendix in the section 6, we present the different types of solutions of physical interest studied in [7]. In the following part we are interested in interactions between nonautonomous solitons; therefore, we study in detail the particular case without dissipation, i.e., $\gamma = 0$, and a constant frequency $\Omega = \Omega_0$. For this election, the Riccati equation determines a nonlinearity described by a function $R(t)$ that satisfies $\tilde{R}(t) = \text{sech}(\Omega_0 t)$. Soliton solutions of (1) with these coefficients are denominated snake solitons [7]. In the following part we study the interaction of multiple snake solitons derived from multisoliton solutions of the NLS equation. The methodology used in this work is general and could be implemented to study other solutions of (10) that include dissipation and diverse types of nonlinearities (see Table 1).

3. Brigth multisoliton solutions of the nonlinear Schrödinger equation

In order to investigate soliton interactions of the GP equation (1), in this section we study the one-dimensional nonlinear Schrödinger equation:

$$i\frac{\partial q(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 q(x,t)}{\partial x^2} + |q(x,t)|^2 q(x,t) = 0. \tag{11}$$

The NLS equation is studied in different contexts in physics and mathematics [1, 8, 9]. This equation is integrable, with soliton solutions product of the compensation between the nonlinearity an the dispersive terms [1]. Solutions with multiple solitons are studied in detail in references [10, 11] by means of the Lax pair formulation and Zakharov-Shabat schemes [1, 9].

N-soliton solutions of equation (11) can be expressed as:

$$q(x,t) = \sum_{k=1}^{N} L_k(x,t), \tag{12}$$

where the functions $L_k(x,t)$ are defined by the algebraic system of equations [10, 11]:

$$L_k(x,t) + \sum_{m=1}^{N}\sum_{n=1}^{N}\eta_{kmn}L_m(x,t)e^{\beta_k+\beta_n^*} = (q_0)_k e^{\beta_k} \qquad \text{for} \qquad k = 1,2,\ldots,N, \tag{13}$$

with the coefficients:

$$\eta_{kmn} = \frac{(q_0)_k (q_0)_n^*}{[(\kappa_m+\kappa_n) + i(\lambda_m-\lambda_n)][(\kappa_k+\kappa_n) + i(\lambda_k-\lambda_n)]}, \tag{14}$$

$$\beta_k = i\left(\frac{\kappa_k^2-\lambda_k^2}{2}t - \lambda_k x\right) - \kappa_k\left[x + \lambda_k t\right]. \tag{15}$$

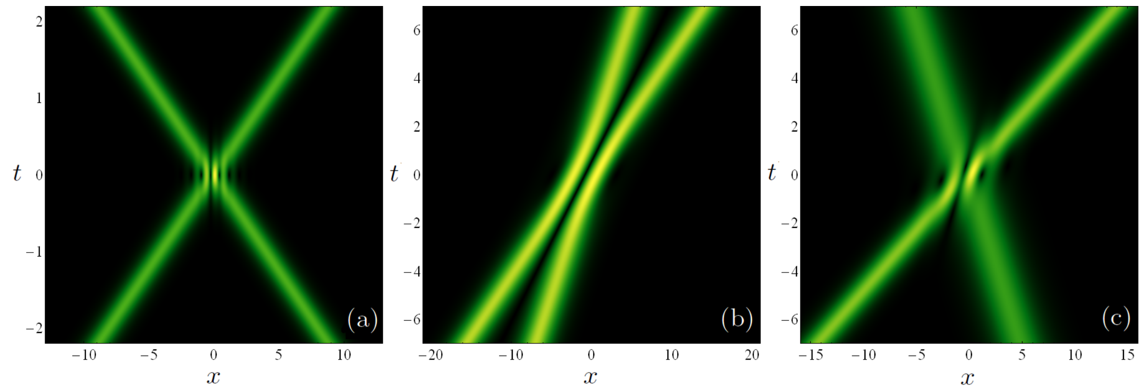

Fig. 1. Density plot of $|q(x,t)|$ as a function of x and t for two-soliton solutions of the NLS equation (11). (a) Frontal collision of solitons with parameters: $\kappa_1 = \kappa_2 = (q_0)_1 = (q_0)_2 = 1.8$ and $\lambda_1 = -\lambda_2 = -4$. (b) Two solitons with parameters: $\kappa_1 = \kappa_2 = 1$, $(q_0)_1 = (q_0)_2 = 2$, $\lambda_1 = -1$, and $\lambda_2 = -2$. (c) Two solitons with parameters: $\kappa_1 = 1.2$, $(q_0)_1 = 2.25$, $\lambda_1 = -1$ for one soliton and $\kappa_2 = 0.8$, $(q_0)_2 = 1.5$, $\lambda_2 = 0.8$ for the other one.

In equations (13)-(15), the set of arbitrary real constants κ_k, λ_k, and complex constants $(q_0)_k$ with $k = 1, 2, .., N$, define characteristics of each soliton.

In the case $N = 1$, it is obtained the soliton solution of the equation (11) that takes the form [1]:

$$q(x,t) = \kappa \frac{q_0}{\sqrt{q_0 q_0^*}} \operatorname{sech}\left[\kappa\left(x + \lambda t\right) - \ln\left|\frac{q_0}{2\kappa}\right|\right] e^{i\left(\frac{\kappa^2 - \lambda^2}{2} t - \lambda x\right)}. \tag{16}$$

Equation (16) shows how each of the constants κ, λ, q_0 are related with dynamical characteristics of the soliton: amplitude, velocity and initial position.

Soliton solutions with $N > 1$ are difficult to write explicitly, however, it is well known that for the interactions between solitons, these solutions are expressed approximately by a superposition of independent solitons and the product of the interaction is a shift of the solitons after the collision [1, 10]. Equations (13)-(15) for $N = 2$ allow to study any case of two soliton interactions. In Figure 1 are presented some particular solutions, for example, in Figure 1(a) is depicted a frontal collision of two pulses with the same amplitude, on the other hand in Figure 1(b) the solution describes a fast soliton that passes a slow soliton with the same amplitude, finally Figure 1(c) shows the dynamics of a frontal collision of solitons with different amplitude. In the case of three solitons are studied collisions of two solitons with equal amplitude and one soliton with greater amplitude. In Figure 2(a), relative velocities between solitons are high so that the solution is in good approximation, the superposition of independent solitons. In Figure 2(b) some of the relative velocities are small and nonlinear effects are significant during the interaction and they produce a shift in the positions of the solitons after the collision.

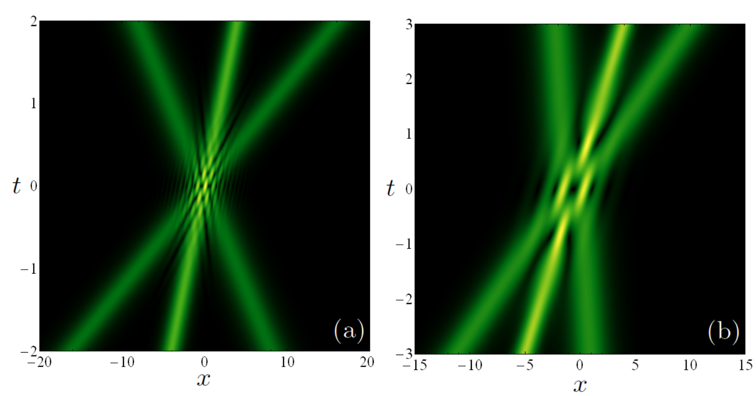

Fig. 2. Density plot of $|q(x,t)|$ as a function of x and t for three-soliton solutions of the NLS equation. Each soliton is defined by the parameters: (a) $\kappa_1 = 2\kappa_2 = 2\kappa_3 = 4$, $(q_0)_1 = 2(q_0)_2 = 2(q_0)_3 = 8$, $\lambda_1 = -2$, $\lambda_2 = -8$ and $\lambda_3 = 4$. (b) $\kappa_1 = 2\kappa_2 = 2\kappa_3 = 4$, $(q_0)_1 = 2(q_0)_2 = 2(q_0)_3 = 8$, $\lambda_1 = -0.5$, $\lambda_2 = -2.5$ and $\lambda_3 = 1.5$.

4. Snake solitons

In the section 2 we have presented a series of manipulations that establish a connection between the one-dimensional Gross-Pitaevskii equation with time-dependent coefficients (1) and the standard nonlinear Schrödinger equation (5). All this analysis is valid when the functions $\tilde{R}(t)$, $\Omega(t)$ are related by the Riccati equation (10). Also, in the section 3 we use a set of algebraic equations (13) to find N-soliton solutions of the nonlinear Schrödinger equation.

In this section we study the particular case $\gamma = 0$, $\Omega = \Omega_0$ constant, and $\tilde{R}(t) = \text{sech}(\Omega_0 t)$ that describes a condensate in a harmonic trap with a constant frequency Ω_0 and a nonlinear term that is modulated in time. The resulting nonlinear differential equation has nonautonomous soliton solutions denominated snake solitons due to the oscillations of the center of mass of the soliton in the trap resembles the movement of a snake [7]. In the following we use the multisoliton solutions of the nonlinear Schrödinger equation to study analytically different types snake soliton collision in a harmonic trap.

Now we obtain two snake soliton solutions of the equation (1) from the solutions of the NLS equation plotted in Figure 1. From the frontal collision in 1(a) is depicted the frontal collision of snake solitons presented in Figure 4, the results are approximately described by independent nonautonomous solitons due to the short time of the interaction. On the other hand, in in Figure 4 is presented a two-soliton solution with a slow interaction, in the interaction

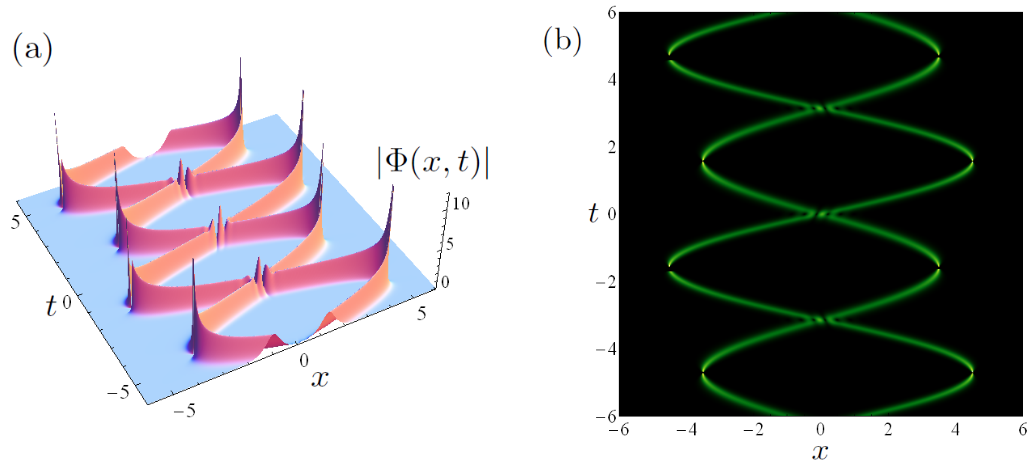

Fig. 3. (a) Dynamics and (b) density plot of $|\Phi(x,t)|$ for the solution of the GP equation (1) with two snake solitons. The solution is obtained from (4), (6)-(9) and the two-soliton solution of the NLS equation plotted in Figure 1(a). Other parameters are: $\Omega_0 = 1$, $r_0 = 1$ and $b = 0.25$.

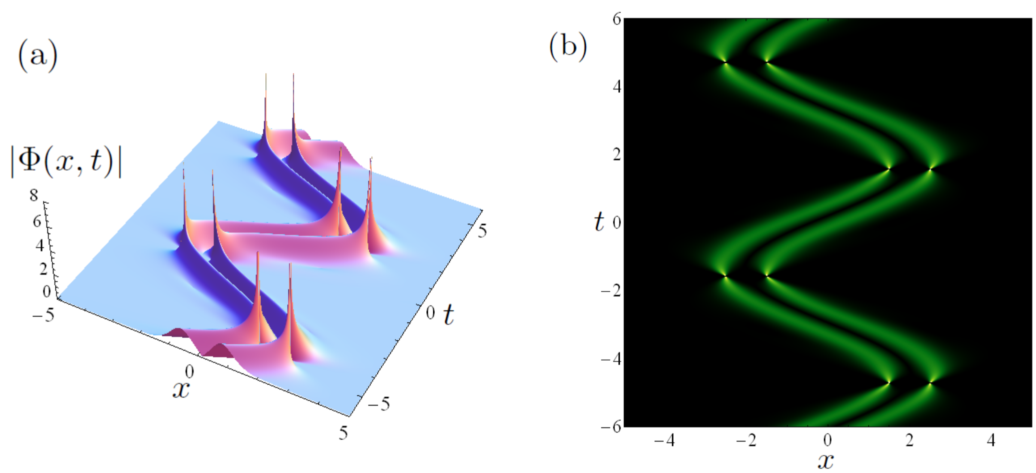

Fig. 4. (a) Dynamics and (b) density plot of $|\Phi(x,t)|$ for the solution of the GP equation (1) with two snake solitons. The solution is obtained from 4, (6)-(9) and the two-soliton solution of the NLS equation plotted in Figure 1(b). Other parameters are: $\Omega_0 = 1$, $r_0 = 1$ and $b = 0.25$.

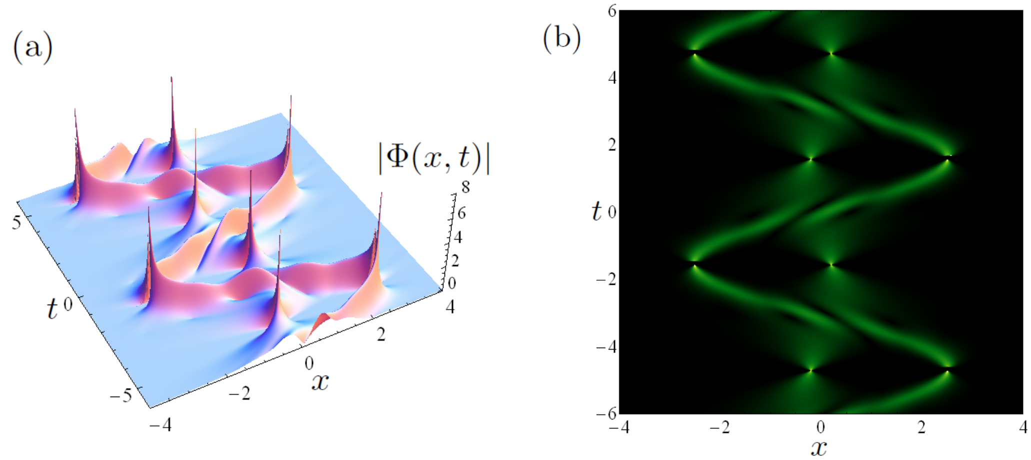

Fig. 5. (a) Dynamics and (b) density plot of $|\Phi(x,t)|$ for the solution of the GP equation (1) with two snake solitons. The solution is obtained from the two soliton solution of the NLS equation plotted in Figure 1(c). Other parameters are: $\Omega_0 = 1$, $r_0 = 1$ and $b = 0.25$.

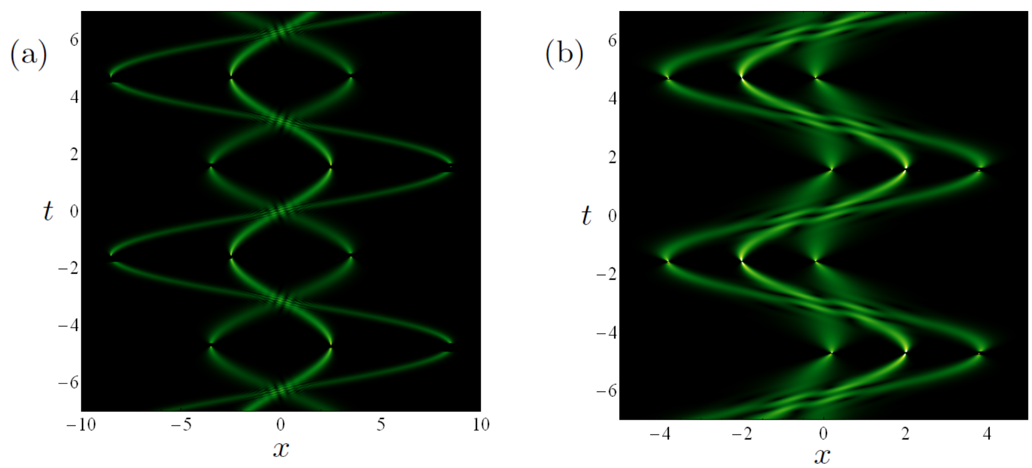

Fig. 6. $|\Phi(x,t)|$ for the solution of the GP equation (1) with three snake solitons. The solution is obtained from the three-soliton solution of the NLS equation plotted in Figure 2. Other parameters are: $\Omega_0 = 1$, $r_0 = 1$ and $b = 0.25$.

the solution is different of the sum of independent solitons. This state has distinct properties of each individual component, in some contexts these types of solutions are denominated soliton molecules [12, 13]. A similar result is obtained in Figure 5 for the interaction of solitons with different amplitude and velocity.

Also, we obtain analytically three snake soliton solutions for the equation (1). In Figure 6 we present the results deduced from the solutions presented in Figure 2. In Figure 6(a), the collisions are fast and as consequence the solutions are well described by the sum of independent solitons. Otherwise, in Figure 6(b) the solutions interact in a nontrivial way and the effects on the small soliton are considerable.

5. Remarks and conclusions

In summary, we have used multisoliton solutions of the nonlinear Schrödinger equation to obtain solutions of the 1D Gross-Pitaevskii equation with time-dependent coefficients that are related by means of a Riccati type differential equation. Although we have restrict our attention to a few selected examples for snake solitons with two and three nonautonomous soliton solutions, the range of cases with physical relevance is very wide. The methodology used is general and could be implemented to study solutions of cases that include dissipation and other types of nonlinearities.

6. Appendix: Solutions of the Riccati equation

There are numerous forms of $\Omega^2(t)$ for which explicit solutions of $\tilde{R}(t)$ can be obtained [7]. The most important of them and with physical relevance are given in the Table 1.

	Form of $\Omega^2(t)$	Physically interesting solution of $\tilde{R}(t)$
1.	Ω_0^2 = constant	$\sec(\Omega_0 t)$
2.	$-\Omega_0^2$ = constant	$\text{sech}(\Omega_0 t),\ \exp(\pm\Omega_0 t)$
3.	$-\frac{\Omega_0^2}{2}\left[1-\tanh\left(\frac{\Omega_0}{2}t\right)\right]$	$1+\tanh\left(\frac{\Omega_0}{2}t\right)$
4.	$-\Omega_0\left(\frac{\Omega_0}{2}\left[1-\cos(2\lambda t)\right]+\lambda\cos(\lambda t)\right)$	$\exp\left[\frac{-\Omega_0}{\lambda}\cos(\lambda t)\right]$
5.	$\Omega_0\lambda\exp(\lambda t)-\Omega_0^2\exp(2\lambda t)$	$\exp\left(\frac{\Omega_0}{\lambda}\exp(\lambda t)\right)$
6.	$3\Omega_0-\Omega_0^2 t^2$	$\frac{1}{t}\exp\left(\frac{\Omega_0 t^2}{2}\right)$
7.	$\Omega_0-\Omega_0^2 t^2$	$\exp\left(\frac{\Omega_0 t^2}{2}\right)$
8.	$\Omega_0 n t^{n-1}-\Omega_0^2 t^{2n}$	$\exp\left(\frac{\Omega_0 t^{n+1}}{n+1}\right)$
9.	$\frac{-2}{t^2}$	$\frac{3t}{t^3+3}$
10.	$\frac{\tilde{b}}{t^2}, \tilde{b}<0$	$\frac{(2\lambda+1)t^\lambda}{(2\lambda+1)+t^{2\lambda+1}}$ with $\lambda=\frac{-1\pm\sqrt{1-4\tilde{b}}}{2}$
11.	$\Omega_0 t\lambda-\Omega_0(\Omega_0+\lambda)\coth^2(\lambda t)$	$[\sinh(\lambda t)]^{\Omega_0/\lambda}$
12.	$\Omega_0 t^2(1-2\coth^2(\Omega_0 t))$	$\sinh(\Omega_0 t)$
13.	$3\Omega_0 t\lambda-\lambda^2-\Omega_0(\Omega_0+\lambda)\tanh^2(\lambda t)$	$\frac{[\cosh(\lambda t)]^{\Omega_0/\lambda}}{\sinh(\lambda t)}$
14.	$2\Omega_0^2\,\text{sech}^2(\Omega_0 t)$	$\coth(\Omega_0 t)$
15.	$3\Omega_0\lambda-\lambda^2-\Omega_0(\Omega_0+\lambda)\coth^2(\lambda t)$	$\frac{[\sinh(\lambda t)]^{\Omega_0/\lambda}}{\cosh(\lambda t)}$
16.	$-2\Omega_0^2\text{csech}^2(\Omega_0 t)$	$\tanh(\Omega_0 t)$
17.	$\Omega_0\lambda-\Omega_0(a+\lambda)\tanh^2(\lambda t)$	$[\cosh(\lambda t)]^{\Omega_0/\lambda}$
18.	$\Omega_0^2(1-2\tanh^2(\Omega_0 t))$	$\cosh(\Omega_0 t)$
19.	$-\Omega_0^2+\Omega_0\lambda\sinh(\lambda t)-\Omega_0^2\sinh^2(\lambda t)$	$\exp\left(\frac{\Omega_0}{\lambda}\sinh(\lambda t)\right)$
20.	$-2\Omega_0^2\left[\tanh^2(\Omega_0 t)+\coth^2(\Omega_0 t)\right]$	$\cosh(\Omega_0 t)\sinh(\Omega_0 t)$
21.	$-\Omega_0^2+\Omega_0\lambda\cos(\lambda t)+\Omega_0^2\cos^2(\lambda t)$	$\exp\left(\frac{-\Omega_0}{\lambda}\cos(\lambda t)\right)$
22.	$-\Omega_0^2+\Omega_0\lambda\sin(\lambda t)+\Omega_0^2\sin^2(\lambda t)$	$\exp\left(\frac{-\Omega_0}{\lambda}\sin(\lambda t)\right)$
23.	$\Omega_0\lambda+\Omega_0(\lambda-\Omega_0)\tan^2(\lambda t)$	$\sec(\Omega_0 t)^{\Omega_0/\lambda}$
24.	$\lambda^2+3\Omega_0\lambda+\Omega_0(\lambda-\Omega_0)\tan^2(\lambda t)$	$\frac{\sec(\Omega_0 t)^{\Omega_0/\lambda}}{\sin(\lambda t)}$
25.	$\lambda^2+3\Omega_0\lambda+\Omega_0(\lambda-\Omega_0)\cot^2(\lambda t)$	$\frac{\sec(\Omega_0 t)}{\sin(\lambda t)^{\Omega_0/\lambda}}$
26.	$\Omega_0\lambda+\Omega_0(\lambda-\Omega_0)\cot^2(\lambda t)$	$\frac{1}{\sin(\lambda t)^{\Omega_0/\lambda}}$
27.	$-2\Omega_0^2\left[\tan^2(\Omega_0 t)+\cot^2(\Omega_0 t)\right]$	$\cos(\Omega_0 t)\sin(\Omega_0 t)$

Table 1
Explicit solutions of the Riccati equation (10) for different forms of $\Omega^2(t)$. The physical context and the implementation of some of these solutions are discussed in the Reference [7].

7. Acknowledgements

The authors are grateful to the Universidad de Nariño for financial support for the development of this work.

References

[1] P.G. Drazin and R.S. Johnson. *Solitons: An Introduction*. Cambridge Computer Science Texts. Cambridge University Press, 1989.

[2] T. Dauxois and M. Peyrard. *Physics of Solitons*. Cambridge University Press, 2006.

[3] E. Infeld and G. Rowlands. *Nonlinear Waves, Solitons and Chaos*. Cambridge University Press, 2000.

[4] V. N. Serkin, Akira Hasegawa, and T. L. Belyaeva. Nonautonomous solitons in external potentials. *Phys. Rev. Lett.*, 98:074102, Feb 2007.

[5] M. Gurses. Integrable Nonautonomous Nonlinear Schrodinger Equations. *ArXiv e-prints*, April 2007.

[6] Anjan Kundu. Integrable nonautonomous nonlinear schrödinger equations are equivalent to the standard autonomous equation. *Phys. Rev. E*, 79:015601, Jan 2009.

[7] S. Rajendran, P. Muruganandam, and M. Lakshmanan. Bright and dark solitons in a quasi-1d bose-einstein condensates modelled by 1d gross-pitaevskii equation with time-dependent parameters. *Physica D: Nonlinear Phenomena*, 239(7):366 – 386, 2010.

[8] A.C. Newell and J.V. Moloney. *Nonlinear optics*. Advanced topics in the interdisciplinary mathematical sciences. Addison-Wesley, 1992.

[9] X.F. Pang and Y.P. Feng. *Quantum Mechanics in Nonlinear Systems*. World Scientific, 2005.

[10] A. Pérez-Riascos and A. Rugeles-Pérez. Soluciones multisolitónicas de la ecuación vectorial no lineal de Schrödinger. *Thesis, Universidad de Nariño*, 2006.

[11] A. Pérez-Riascos and A. Rugeles-Pérez. Soluciones multisolitónicas de la ecuación NLS. *Revista Colombiana de Física*, 38:585–588, 2006.

[12] U Al Khawaja and H T C Stoof. Formation of matter-wave soliton molecules. *New Journal of Physics*, 13(8):085003, 2011.

[13] Kazimierz Łakomy, Rejish Nath, and Luis Santos. Soliton molecules in dipolar bose-einstein condensates. *Phys. Rev. A*, 86:013610, Jul 2012.