

Transition Amplitude to Podolsky's Electromagnetic Theory

Amplitud de Transición para la Teoría Electromagnética de Podolsky

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Resumen

En este trabajo se cuantizará la teoría electromagnética de Podolsky utilizando la formulación de integrales de trayectoria y la aproximación de Dirac para determinar la estructura de ligaduras de la teoría. El método de Faddeev-Senjanovic permitirá calcular la amplitud de transición de la que derivaremos el propagador del campo.

Palabras Claves: Teoría Electromagnética de Podolsky, Método de Dirac, Amplitud de Transición vacío-vacío, Propagador del Fotón.

Abstract

In this work we quantize the Podolsky's Electromagnetic Theory using the path integral formalism following the Dirac's approach to determine the constraint structure. The Faddeev-Senjanovic method will determine the transition amplitude from which we derive the field propagator.

Keywords: Podolsky's Electromagnetic Theory, Dirac Method, Vacuum-Vacuum transition, Photon Propagator.

1. Introduction

Field theories with higher order derivative Lagrangians have been studied actively in the past [1]. This kind of theories have been used to solve the problem of renormalization of the gravitational field by inserting a quadratic term of the scalar curvature to the Einstein-Hilbert lagrangian [2]. Recently, higher order derivative Lagrangians have been used as a method for regularization of the ultraviolet divergences of gauge invariance supersymmetric theories [3].

Effective models in gauge theories were proposed through the possibility of using higher order derivative Lagrangians. Yang-Mills can be approximated, at the limit of strong coupling, by an effective lagrangian containing the second derivative of the field strength tensor [4]. From the possibility that the gluon propagator could have an infrared asymptotic behaviour, was proposed an effective Lagrangian containing a cubic term in the field strength tensor and a quadratic term in the first covariant derivative of the same tensor [5].

In fact, one of the first attempts to use higher order lagrangian dates back to work from Boop, Podolsky and Schwed [6] who attempt to modify the Maxwell's electrodynamics to get rid of the infinities of the theory such as the electron self-energy and the vacuum polarization current. In the non-quantum case this difficulties has been overcome by adding to the Maxwell's Lagrangian a quadratic term in the divergence of the field strength tensor. The above theory

has many interesting features already at the classical level. It gives the correct expression for the self-force of charged particles at short distances [7], the theory also preserves invariance under $U(1)$, and yields field equations that are still linear in the fields. It was showed that the Podolsky's lagrangian is the unique linear second order generalization from Maxwell-Lorentz theory and the most general one for the $U(1)$ gauge group [8]. The important prediction of the model is the existence of massive photons, where the mass is proportional to the inverse of the Podolsky's parameter a , which allow that experiments may test the generalized electrodynamics as a viable effective theory.

The canonical quantization of the theory was tried in the paper of Podolsky and Schwed [6]. However, Podolsky's theory suffer the same difficulties of the standard electromagnetic field, the presence of a degenerate variable, which had forced them to use a Fermi-like Lagrangian. Moreover, the chosen gauge fixing, the usual Lorenz condition, does not fulfill the requirements for a good gauge choice in the context of Podolsky's theory. The first consistent approach to the quantization of the theory was given by Galvão and Pimentel [9], where they analyzed the generalized electrodynamics from the Hamiltonian point of view, using the Dirac's theory for constrained systems [10]. The problem of gauge fixing for the theory was studied in detail and the correct generalization of the radiation gauge was obtained and the Dirac Brackets (DB) for the dynamical variables in this gauge were calculated.

The present work is addressed to study the functional quantization of the Podolsky's electromagnetic theory. The paper is organized as follow. In the Sect. 2, we present a brief review of the dynamics of the theory. In Sect. 3, we present the analysis of the canonical structure of the theory. In Sect. 4, we calculate the corresponding covariant vacuum-vacuum amplitude and derive the photon propagator. At the end, we present our conclusions.

2. Dynamics of the Podolsky theory

The Podolsky's electromagnetic theory is based on the following lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a^2}{2}\partial_\lambda F^{\alpha\lambda}\partial^\rho F_{\alpha\rho} \quad (1)$$

where the field-strength tensor is expressed in terms of the potential in the usual way, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, a is a constant with the dimensions of length. The above lagrangian reduces to Maxwell theory when $a = 0$. The Euler-Lagrange equations of motion follow from Hamilton's principle, $\delta S = \delta \int_\Sigma d^4x \mathcal{L} = 0$, with $\delta x^\mu = 0$ and $\delta A^\mu|_{\partial\Sigma} = 0$, where $\partial\Sigma$ is the boundary of Σ and are given by

$$\mathbf{L}[A_\theta] = \frac{\partial\mathcal{L}}{\partial A_\theta} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\theta)} + \partial_\mu \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu \partial_\nu A_\theta)} = (1 + a^2\Box) \partial_\mu F^{\mu\theta} = 0 \quad (2)$$

with $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. Podolsky Lagrangian does not lead to the equations of motion expected from the Maxwell theory, therefore, they are non-equivalent descriptions of the Abelian gauge field. Defining the electric and magnetic field by $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$ respectively, the lagrangian assumes the form

$$\mathcal{L} = \frac{1}{4}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{a^2}{2}\left[(\nabla \cdot \mathbf{E})^2 - (\dot{\mathbf{E}} - \nabla \times \mathbf{B})^2\right], \quad (3)$$

while the equation of motion are written as

$$(1 + a^2\Box) \nabla \cdot \mathbf{E} = 0 \quad , \quad (1 + a^2\Box) (\dot{\mathbf{E}} - \nabla \times \mathbf{B}) = 0. \quad (4)$$

The symmetric Energy-Momentum density tensor reads [11]:

$$\begin{aligned} \mathcal{T}^{\mu\nu} = & F^\mu{}_\lambda F^{\lambda\nu} - \eta^{\mu\nu} \mathcal{L} + a^2 (2\partial^\lambda F^{\mu\xi} \partial_\lambda F^\nu{}_\xi + \\ & - 2\partial^\lambda F^{\xi\mu} \partial_\xi F^\nu{}_\lambda + \partial_\lambda F^{\lambda\mu} \partial_\xi F^{\xi\nu}). \end{aligned} \quad (5)$$

The energy density \mathcal{E} is the component \mathcal{T}^{00} of this tensor. It is possible to write \mathcal{E} in terms of the electric and magnetic fields:

$$\mathcal{E} = \frac{1}{2} \left\{ \mathbf{E}^2 + \mathbf{B}^2 + a^2 \left[(\nabla \cdot \mathbf{E})^2 + (\dot{\mathbf{E}} - \nabla \times \mathbf{B})^2 + 4\mathbf{E} \cdot \Box \mathbf{E} + 4\mathbf{E} \cdot \nabla (\nabla \cdot \mathbf{E}) \right] \right\}. \quad (6)$$

This expression appears to not be positive-definite in the general case. However, if we restrict it to the electrostatic case, we have for the energy:

$$E_{electrost.} = \int d^3x \mathcal{E}_{electrost.} = \frac{1}{2} \int d^3x [\mathbf{E}^2 + a^2 (\nabla \cdot \mathbf{E})^2]. \quad (7)$$

Once we impose the condition $E_{electrost.} \geq 0$, we have the implication that the parameter a must be real and, without loss of generality, we assume it to be positive. For a point charge the electrostatic potential is given by

$$\varphi(r) = \frac{e}{r} (1 - e^{-\frac{r}{a}}) \quad (8)$$

and it is possible to prove that the total energy has a finite value equal to $\frac{e^2}{2a}$, this being a remarkable result. The expression for $\varphi(r)$ show the presence of a Yukawa-type potential besides the usual Coulomb potential.

3. Constraints structure

In passing to Hamiltonian formalism it is possible to show that the lagrangian which describe the theory is singular [10]. The canonical momenta p^μ and π^μ conjugate to A_μ and ϕ_μ respectively, where $\phi_\mu \equiv \partial_0 A_\mu$ is considered as an independent variable, are defined by [9]:

$$p^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\mu} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_\mu)} - 2 \partial_k \frac{\partial \mathcal{L}}{\partial (\partial_k \phi_\mu)} = F^{\mu 0} - a^2 [\eta^{\mu k} \partial_k \partial_\lambda F^{0\lambda} - \partial_0 \partial_\lambda F^{\mu\lambda}] \quad (9)$$

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_\mu)} = a^2 [\eta^{\mu 0} \partial_\lambda F^{0\lambda} - \partial_\lambda F^{\mu\lambda}]$$

with the above expressions we get the primary constraints

$$\Omega_1 \equiv \pi^0 \approx 0 \quad , \quad \Omega_2 \equiv p^0 - \partial_k \pi^k \approx 0 \quad (10)$$

Following the Dirac's procedure we define the canonical Hamiltonian density, which to higher order derivative is defined by

$$\mathcal{H}_C \equiv p^\mu \dot{A}_\mu + \pi^\mu \dot{\phi}_\mu - \mathcal{L} \quad (11)$$

$$= p^\mu \phi_\mu + \frac{1}{4a^2} (\pi^k)^2 + \pi^k (\partial_k \phi_0 - \partial_l F_{kl}) - \frac{1}{2} (\phi_k - \partial_k A_0)^2 + \frac{1}{4} (F_{ki})^2 - \frac{a^2}{2} (\partial_k \phi_k - \partial_k \partial_k A_0)^2 \quad (12)$$

now, we add an arbitrary linear combination of the primary constraints (10) to the canonical hamiltonian to obtain

$$H_P = \int d^3y [\mathcal{H}_C + u^1(y) \Omega_1(y) + u^2(y) \Omega_2(y)] \quad (13)$$

here u^i are Lagrange multipliers. The relation (13) is defined as the primary Hamiltonian and the Dirac's procedure tell us that the primary constraints must be preserved in the time under time evolution generated by the primary Hamiltonian by requiring that they have a weakly vanishing PB with the H_P , $(\dot{\Omega}_i = \{\Omega_i, H_C\} \approx 0, i = 1, 2)$. With the fundamental Poisson brackets (PB) defined by

$$\{A_\mu(x), p^\nu(y)\} = \delta_\mu^\nu \delta^3(x-y) \quad , \quad \{\phi_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta^3(x-y), \quad (14)$$

such requirement yields

$$\dot{\Omega}_1 = -\Omega_2 \approx 0 \quad , \quad \dot{\Omega}_2 = \partial_k p^k \equiv \Omega_3(x) \approx 0 \quad (15)$$

so that there is a secondary constraint where $\dot{\Omega}_3 \approx 0$. the set of constraints $\Omega \approx 0, i = 1, 2, 3$ are clearly first class [10] and no more constraints are generated. Finally we can write the extended Hamiltonian as

$$H_E = \int d^3y [\mathcal{H}_C + w^a(y) \Omega_a(y)] \quad , \quad a = 1, 2, 3. \quad (16)$$

this is the Hamiltonian that generates the time evolution of the system with full gauge freedom. We analyse the Hamiltonian equations of motion, thus the dynamics of the fields is given by

$$\partial_0 A_\mu = \delta_\mu^0 (\phi_0 + w^2) + \delta_\mu^k (\phi_k - \partial_k w^3) \quad (17)$$

$$\partial_0 \phi_\mu = \delta_\mu^0 w^1 + \delta_\mu^k \left[-\frac{1}{a^2} \pi^k + \partial_k \phi_0 - \partial_n F_{kn} + \partial_k w^2 \right]$$

and for the canonical momenta

$$\begin{aligned} \partial_0 p^\mu &= \delta_0^\mu [\partial_m (\phi_m - \partial_m A_0) - a^2 \nabla^2 (\partial_m \phi_m - \nabla^2 A_0)] + \delta_k^\mu [\partial_m F_{mk} + \partial_k \partial_m \pi^m - \nabla^2 \pi^k], \\ \partial_0 \pi^\mu &= \delta_0^\mu [-p^0 + \partial_m \pi^m] + \delta_k^\mu [-p^k + \phi_k - \partial_k A_0 - a^2 \partial_k (\partial_m \phi_m - \nabla^2 A_0)]. \end{aligned} \quad (18)$$

from (17) and (18) it is easy to obtain

$$(1 + a^2 \square) \partial_\lambda F^{\lambda\mu} \approx -\delta_0^\mu (1 - a^2 \nabla^2) \nabla^2 w^3 \quad (19)$$

thus, the equation of motions are consistent with its lagrangian form (2) if we chose $w^3 = 0$.

The Dirac's algorithm requires as many gauge conditions as first class constraints there are. However, such gauge fixing conditions must be compatible with the Euler-Lagrange equations and therefore they must fix the Lagrange multipliers w^a and with together the first class constraints must be a second class set. Galvão and Pimentel showed that a set of appropriated non-covariant canonical gauge conditions which allow to fix the first class constraint are [9]

$$\begin{aligned} \Sigma_1 &\equiv A_0 \approx 0 \\ \Sigma_2 &\equiv \phi_0 \approx 0 \\ \Sigma_3 &\equiv (1 + a^2 \square) \partial_k A_k \approx 0 \end{aligned} \quad (20)$$

4. Path integral quantization

Now, we will construct the transition amplitude for Podolsky's electromagnetic theory. The path integral quantization is accomplished according to Faddeev-Senjanovic method [12], by extending their expression for the partition function to higher order theories. We have to pay attention to the fact that $\phi_\mu \equiv \partial_0 A_\alpha$ is now an independent canonical variable, and consequently, it has also to be functionally integrated [13]. Thus, the expression of the transition amplitude can be written in the following way

$$Z = \int D\mu \exp \left\{ i \int d^4x [p^\mu \partial_0 A_\mu + \pi^\mu \partial_0 \phi_\mu - \mathcal{H}_C] \right\}, \quad (21)$$

and \mathcal{H}_C is the canonical hamiltonian is given by (11).The integration measure is defined by

$$D\mu \equiv D\pi^\mu D\phi_\mu Dp^\mu DA_\mu \det |\{\Omega_a, \Sigma_b\}| \delta(\Omega_a) \delta(\Sigma_b) \quad (22)$$

Here, $\det |\{\Omega_\alpha, \Sigma_\beta\}_B|$ represent the determinant formed by the brackets between the first class constraints and the gauge fixing conditions. We can determine that this determinant take the form

$$\det |\{\Omega_\alpha, \Sigma_\beta\}| = \det \left| - (1 + a^2 \nabla^2) \nabla^2 \right|. \quad (23)$$

thus, it does not contain field variables and can be absorbed in a normalization constant.

Introducing (10), (11), (15), (20) and (22) into (21), integrating over the momenta and field variables and using the delta functional, we arrived in the following expression for the transition amplitude

$$Z = \int DA_\mu \det \left| - (1 + a^2 \nabla^2) \nabla^2 \right| \delta \left[(1 + a^2 \square) \partial^k A_k \right] \exp \left\{ i \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial^\mu F_{\mu\beta} \partial_\alpha F^{\alpha\beta} \right] \right\}. \quad (24)$$

However, the last expression is non-covariant, thus, for calculation purposes we can use the ansatz of Faddeev-Popov-De Witt [14] to get a covariant expression for the transition amplitude. Like pointed by Pimentel and Galvão [9] the usual Lorenz gauge condition, $\partial^\mu A_\mu = 0$, is not a suitable gauge choice because it does not satisfy a certain number

of requirements necessary to be a good gauge condition for the theory. They showed that the generalized Lorenz gauge condition

$$f = (1 + a^2 \square) \partial^\mu A_\mu. \quad (25)$$

satisfy all the requirements stated [9] and this choice of gauge is as natural in Podolsky's theory as the Lorentz gauge is in Maxwell's electrodynamics. So, in this way, we obtain the desired covariant vacuum-vacuum transition amplitude

$$Z = \int DA_\mu \det \left| - (1 + a^2 \square) \square \right| \delta [f - (1 - 2a^2 \square) \partial_\mu A^\mu] \exp \left\{ i \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial^\mu F_{\mu\beta} \partial_\alpha F^{\alpha\beta} \right] \right\}. \quad (26)$$

Where the generating functional is independent of $f(x)$ we can integrate in $f(x)$ with weight $\exp \left(-\frac{i}{2\xi} \int d^4x f^2(x) \right)$, thus we obtain

$$Z = \int DA_\mu \det \left| - (1 + a^2 \square) \square \right| \exp \{ i S_\xi [A_\mu] \} \quad (27)$$

with

$$S_\xi [A_\mu] \equiv \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu F^{\mu\alpha} \partial_\nu F^\nu_\alpha - \frac{1}{2\xi} [(1 + a^2 \square) \partial_\mu A^\mu]^2 \right] \quad (28)$$

In this covariant gauge choice we see that the Faddeev-Popov-De Witt determinant not contain field variables (the ghost are uncoupling with gauge fields) and so, it also can be absorbed in a normalization constant N .

Next let us define the generating functional

$$Z [J_\mu] = N \int DA_\mu \exp [i S_{eff}], \quad (29)$$

where

$$S_{eff} = \int d^4x \left[-\frac{1}{2} A_\mu P^{\mu\nu} A_\nu + A^\mu J_\mu \right] \quad (30)$$

and with $P^{\mu\nu}$ defined by

$$P^{\mu\nu} \equiv -\square g^{\mu\nu} + a^2 \square \left[-\square g^{\mu\nu} - (1 + a^2 \square) \partial^\mu \partial^\nu \right]. \quad (31)$$

Here J_μ are the sources associated to the photon field. We have took $\xi = 1$ which represent the generalized Feynman gauge condition. From $Z [J_\mu]$, the Feynman propagator associated with A_μ is obtained by functional differentiation of the following way

$$\left\langle T \hat{A}_\mu(x) \hat{A}_\nu(x) \right\rangle = -\frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(x)} Z [J_\mu], \quad (32)$$

where T denote time ordering. After a laborious calculus, is possible to show that the Photon propagator is

$$D_{\mu\nu}(x, y) = \frac{g_{\mu\nu}}{\square_x} \delta^4(x - y) - \left\{ g_{\mu\nu} + \left[\frac{1}{\square_x} - \frac{1}{\square_x + \frac{1}{a^2}} \right] \partial_\mu^x \partial_\nu^x \right\} \frac{1}{(\square_x + \frac{1}{a^2})} \delta^4(x - y) \quad (33)$$

which in the momentum space can be write

$$P_{\mu\nu}(k) = -\frac{i}{k^2} g_{\mu\nu} + i \left\{ g_{\mu\nu} + \left[\frac{1}{k^2} - \frac{1}{k^2 - \frac{1}{a^2}} \right] k_\mu k_\nu \right\} \frac{1}{(k^2 - \frac{1}{a^2})} \quad (34)$$

5. Remarks and conclusions

In this work we have quantized the Podolsky's electromagnetic theory. The covariant vacuum-vacuum transition amplitude was derived in the generalized Lorenz gauge condition. We observed that the Faddeev-Popov-De Witt determinant did not contain field variables, thus, the ghost are uncoupling with the gauge field, and the determinant can be absorbed in a normalization constant. From the generating functional, the Feynman propagator associated with photon field was derived and we observed that in addition of the massless photon there is massive contribution, where the mass is proportional to the inverse of the Podolsky's parameter: $m = \frac{1}{a}$.

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