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Stability tests for dynamical systems and their applications

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Abstract. In this paper we present two stability tests for systems of ordinary differential equations, the first test is for global stability while the second is for local stability. In addition, we use stability tests to analyze both the local and global stability of a nonlinear system of ordinary differential equations.

Keywords. Dynamical systems, Equilibrium solution, Stability.

Resumen. En este artículo se presentan dos criterios de estabilidad para sistemas de ecuaciones diferenciales ordinarias, el primer criterio es para estabilidad global mientras que el segundo es para estabilidad local. Además, los utilizamos para analizar la estabilidad local y global de un sistema no lineal de ecuaciones diferenciales ordinarias.

Palabras Clave. Sistemas dinámicos, Soluciones de equilibrio, Estabilidad.

1. Introduction

Analytical solutions of the nonlinear system of autonomous ordinary differential equations

$$\dot{x} = f(x), \tag{1.1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a $\mathcal{C}^1$ map and $\mathcal{D} \subset \mathbb{R}^n$ is an open set, can not always be explicitly determined. In 1892, A. M. Lyapunov developed his stability theory for nonlinear ordinary differential equations which characterizes the behavior of the dynamical systems trajectories in the sense that nearby solutions remain that way from now on (Hirsch and Smale, [9]). He established the *direct and indirect methods of Lyapunov* (DML and IML), very useful stability criteria for system (1.1). Currently, these methods are the most used to perform theoretical analysis of the stability of equilibrium solutions of (1.1). The main setback of the DML is find a Lyapunov function, because there is not a systematic method for finding. In the IML, determining the sign of the real part of the eigenvalues of the Jacobian matrix evaluated in equilibrium solutions is not always an easy task. For the above reasons, criteria that do not involve the explicit formulation of lyapunov functions or avoid determining the

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sign of the real part of the eigenvalues are more practical to perform the stability analysis. In this paper we present two stability criteria developed by E. Ibarguen et al. in [11,12], and we use them to analyze the stability of several dynamical systems.

2. First test of stability

In this section we establish a test for the asymptotic stability of the system (1.1) equilibrium when $\mathcal{D}$ is an open subset of

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

The following proposition relates the equilibrium stability with the sign of certain determinants.

Proposition 2.1. *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^n_+$ containing $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. Suppose that the function $f : \mathcal{D} \to \mathbb{R}^n$ defined in (1.1) satisfies $f \in \mathcal{C}^1(\mathcal{D})$ and $f(\bar{x}) = 0$. Let $\Delta_j(\bar{x})$ be the determinants defined by*

$$\Delta_j(\bar{x}) = (-1)^j \left| \frac{a_j}{\bar{x}_j} \frac{\partial f_j(\bar{x})}{\partial x_i} + \frac{a_i}{\bar{x}_i} \frac{\partial f_i(\bar{x})}{\partial x_j} \right|_{i=1,\ldots,j}, \quad j = 1, \ldots, n \tag{2.1}$$

where a_j is a positive constant.

1. *If $\Delta_j(\bar{x})$ for $j = 1, \ldots, n$ are positive, then $\bar{x}$ is globally asymptotically stable.*
2. *If $\Delta_j(\bar{x})$ for $j = 1, \ldots, n$ has alternate signs starting with a negative value, then $\bar{x}$ is unstable.*

See [11] for a proof of Proposition 2.1.

3. Second test of stability

The following result focus on determining conditions that allow us to establish the sign of the eigenvalues of $Df(\bar{x})$ through the Gershgorin circles (See, [12]).

Proposition 3.1. *Let $\bar{x}$ an equilibrium point of (1.1),*

$$Df(\bar{x}) = \begin{pmatrix} J_{11} & J_{12} & \cdots & J_{1n} \\ J_{21} & J_{22} & \cdots & J_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n1} & J_{n2} & \cdots & J_{nn} \end{pmatrix}, \tag{3.1}$$

the Jacobian matrix of (1.1) evaluated in $\bar{x}$ and

$$R_i = \sum_{j=1, j\neq i}^{n} |J_{ij}|, \tag{3.2}$$

for $i = 1, \ldots, n$. If $J_{ii} < 0$ and $R_i < |J_{ii}|$ for $i = 1, \ldots, n$ then $\bar{x}$ is locally asymptotically stable.

See [12] for a proof of Proposition 3.1. The following result is a consequence of the Proposition 3.1.

Corolary 3.2. *Let $\bar{x}$ an equilibrium point of (1.1), $Df(\bar{x})$ defined in (3.1), and*

$$R_j = \sum_{i=1, i\neq j}^{n} |J_{ij}|, \tag{3.3}$$

for $j = 1, \ldots, n$. If $J_{jj} < 0$ and $R_j < |J_{jj}|$ for $j = 1, \ldots, n$ then $\bar{x}$ is locally asymptotically stable.

4. Application of stability tests

In this section we apply the two tests formulated in Section 3 to analyze the stability of the following system of ordinary differential equation

$$\frac{dx_j}{dt} = \alpha_j x_j (1 - x_j) - \sigma_j \prod_{i=1}^{n} x_i, \quad j = 1, 2, \ldots, n, \tag{4.1}$$

where $0 < \alpha_j < 1$ and $0 < \sigma_j < 1$ for $j = 1, 2, \ldots, n$.

4.1. Stability analysis by test 1

Let

$$\mathcal{D}_1 = \{x \in \mathbb{R}^n : 0 < x_j < 1, j = 1, 2, \ldots, n\}. \tag{4.2}$$

The following lemma ensures that all solutions of (4.1) starting in the closure of $\mathcal{D}_1$ denoted by $\mathcal{D}_1$, remain there for all $t \geq 0$.

Lema 4.1. *The set $\mathcal{D}_1$ defined in (4.2) is positively invariant for the solutions of the system (4.1).*

See [11] for a proof of Lemma 4.1. The next proposition summarizes existent results of the equilibrium solutions of (4.1).

Proposition 4.2. *The system (4.1) has at least $2^n + 1$ equilibrium solution in $\mathcal{D}_1$.*

To proof the following proposition we use the first test of stability.

Proposition 4.3. *Suppose that the system (4.1) has an interior steady state $\bar{x} \in \mathcal{D}_2 \subset \mathcal{D}_1$ where*

$$\mathcal{D}_2 = \{x \in \mathbb{R}^n : 0 < x_i < 1, 0 < x_i + x_j < 1 \ i, j = 1, 2, \ldots, n\}.$$

then this steady state is globally asymptotically stable on the interior set of $\mathcal{D}_1$.

Proof. From (4.1) we conclude that:

$$f_j(x) = \alpha_j x_j (1 - x_j) - \sigma_j \prod_{k=1}^{n} x_k, \quad j = 1, 2, \ldots, n,$$

which implies that

$$\frac{\partial f_j(\bar{x})}{\partial \bar{x}_j} = \alpha_j (1 - \bar{x}_j) - \alpha_j \bar{x}_j - \sigma_j \prod_{k=1, k\neq j}^{n} \bar{x}_k, \quad j = 1, 2, \ldots, n. \tag{4.3}$$

From equilibrium equations we have:

$$\alpha_j (1 - \bar{x}_j) - \sigma_j \prod_{k=1, k\neq j}^{n} \bar{x}_k = 0, \quad j = 1, 2, \ldots, n. \tag{4.4}$$

Therefore, substituting (4.4) in (4.3) we verify the first hypothesis of Proposition 2.1, that is

$$\frac{\partial f_j(\bar{x})}{\partial \bar{x}_j} = -\alpha_j \bar{x}_j < 0, \quad j = 1, 2, \ldots, n.$$

On the other hand,

$$\begin{aligned} l_{ij}(\bar{x}) &= \left(\frac{\partial f_i(\bar{x})}{\partial x_i}\right)^{-1} \frac{\partial f_i(\bar{x})}{\partial x_j} + \left(\frac{\partial f_j(\bar{x})}{\partial x_j}\right)^{-1} \frac{\partial f_j(\bar{x})}{\partial x_i} \\ &= (-\alpha_i \bar{x}_i)^{-1} \left(-\sigma_i \prod_{k=1, k\neq j}^{n} \bar{x}_k\right) + (-\alpha_j \bar{x}_j)^{-1} \left(-\sigma_j \prod_{k=1, k\neq i}^{n} \bar{x}_k\right) \\ &= \left(\frac{\sigma_i}{\alpha_i} + \frac{\sigma_j}{\alpha_j}\right) \prod_{k=1, k\neq i, k\neq j}^{n} \bar{x}_k. \end{aligned} \tag{4.5}$$

From (4.4) we have:

$$\frac{\sigma_j}{\alpha_j} = \frac{1 - \bar{x}_j}{\prod_{k=1, k\neq j}^{n} \bar{x}_k}, \quad j = 1, 2, \ldots, n. \tag{4.6}$$

Substituting (4.6) in (4.5) we obtain:

$$l_{ij}(\bar{x}) = \frac{1 - \bar{x}_i}{\bar{x}_j} + \frac{1 - \bar{x}_j}{\bar{x}_i}, \quad \text{for } i \neq j.$$

From hypothesis $\bar{x} \in \mathcal{D}_2$, results that $0 < \bar{x}_i + \bar{x}_j < 1$ which implies $(\bar{x}_i + \bar{x}_j)^2 < \bar{x}_i + \bar{x}_j$, or equivalently $(1 - \bar{x}_i)\bar{x}_i + (1 - \bar{x}_j)\bar{x}_j > 2\bar{x}_i\bar{x}_j$. The above implies that the second hypothesis of Proposition 2.1 is satisfied. That is $l_{ij} > 2$. Therefore $\bar{x}$ is globally asymptotically stable on interior set of $\mathcal{D}_1$. ☑

Proposition 2.1 can be use to analyze the local stability for SIR model, SEIR Model, among other [2, 4, 5, 7, 8].

4.2. Stability analysis by test 2

Let $n = 2$, then the system (4.1) is written as

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha_1 x_1(1 - x_1) - \sigma_1 x_1 x_2 \\ \frac{dx_2}{dt} &= \alpha_1 x_2(1 - x_2) - \sigma_2 x_1 x_2. \end{aligned} \tag{4.7}$$

The equilibrium solution of (4.7) are

$$\begin{aligned} E_1 = (0,0) \quad &, \quad E_2 = (1,0) \\ E_3 = (0,1) \quad &, \quad E_4 = (x_1^*, x_2^*), \end{aligned} \tag{4.8}$$

where

$$x_1^* = \frac{1 - \dfrac{\sigma_1}{\alpha_1}}{1 - \dfrac{\sigma_1\sigma_2}{\alpha_1\alpha_2}} \text{ and } x_2^* = \frac{1 - \dfrac{\sigma_2}{\alpha_2}}{1 - \dfrac{\sigma_1\sigma_2}{\alpha_1\alpha_2}}. \tag{4.9}$$

Notice that $0 < x_1^* < 1$ and $0 < x_2^* < 1$. The Jacobian matrix of (4.7) is given by

$$J(x) = \begin{pmatrix} \alpha_1(1 - 2x_1) - \sigma_1 x_2 & -\sigma_1 x_1 \\ -\sigma_2 x_2 & \alpha_2(1 - 2x_2) - \sigma_2 x_1 \end{pmatrix}. \tag{4.10}$$

Evaluating (4.10) in equilibria E_i for $i = 1, \ldots, 4$ we obtain

$$J(E_1) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad , \quad J(E_2) = \begin{pmatrix} -\alpha_1 & -\sigma_1 \\ 0 & \alpha_2 - \sigma_2 \end{pmatrix}$$
$$J(E_3) = \begin{pmatrix} \alpha_1 - \sigma_1 & 0 \\ -\sigma_2 & -\alpha_2 \end{pmatrix} \quad , \quad J(E_4) = \begin{pmatrix} -\alpha_1 x_1^* & -\sigma_1 x_1^* \\ -\alpha_2 x_2^* & -\sigma_2 x_2^* \end{pmatrix}. \tag{4.11}$$

Notice that $J(E_4)$ satisfies the hypothesis $J_{ii} < 0$ for $i = 1, 2$ of Proposition 3.1. To $J(E_3)$ and $J(E_2)$ the above hypothesis is satisfied when $\alpha_1 < \sigma_1$ and $\alpha_2 < \sigma_2$, respectively, and $J(E_1)$ does not satisfy the condition. On the other hand, to $J(E_3)$ and $J(E_2)$ the hypothesis $R_i < |J_{ii}|$ for $j = 1, 2$ is satisfied when $\sigma_2 < \alpha_2$ and $\sigma_1 < \alpha_1$, respectively, to $J(E_4)$ the hypothesis is satisfied when $\alpha_1 > \sigma_1$ and $\alpha_2 > \sigma_2$. Therefore, if $\alpha_1 < \sigma_1$ and $\sigma_2 < \alpha_2$ then E_3 is locally asymptotically stable, if $\alpha_2 < \sigma_2$ and $\sigma_1 < \alpha_1$ then E_2 is locally asymptotically stable, and if $\alpha_1 > \sigma_1$ and $\alpha_2 > \sigma_2$ then E_4 is locally asymptotically stable.
Proposition 3.1 can be use to analyze the global stability of several model [1, 7, 13–31].

5. Discussion

Qualitative analysis of dynamical systems is relevant to determine the behavior of their solutions. The direct and indirect methods of Lyapunov are the most used. However, in a wide range of models, applying them, it is a difficult task to perform. In this sense, efforts have been made to make these criteria much more practical tools to carry out the stability analysis of equilibrium solutions. Other well-known techniques are the next generation matrix method and the Routh-Hurwitz criterion [3, 10, 32]. Stability tests formulated in Sections 2 and 3 are useful in several areas of mathematics and engineering sciences. First test can be use to analyze global stability and the second one to analyze local stability. Both of them are practical tools to determine stability of systems of ordinary differential equations. They are not robust as those mentioned above. However, they are very simple and practical tool to analyze the stability of equilibrium points.

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